# Real Version of Calculus of Complex Variable (II): Cauchy's Point of View*

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**Abstract**

As was the case in a previous paper, the differential form $x+ydxdy$ plays the role $z$ in the standard calculus of complex variable. The role of holomorphic functions will now be played by strict harmonic differential forms in the Kähler algebra of the real plane. These differential forms satisfy the Cauchy-Riemann relations.

No new concept of differentiation is needed, and yet this approach parallels standard Cauchy theory, but more simply. The power series and theorem of residues come here at the end, unlike in the previous paper.

## 1 Introduction

In a previous paper, we continued developing corollary developments of Stokes theorem in multiply connected regions of the real plane. Through the use of Kähler algebra (Clifford algebra of differential forms), it was found that one does not need the calculus of complex variable in order to handle the integrations for which a physicist uses it. The focus was on closed differential 1-forms, functions with power expansions being of the essence very early in the argument. The first major result was the theorem of residues.

We shall now give a different version of real calculus to replace Cauchy's theory. The focus will be on Even DIifferential Forms ("edifs", $u + vdxdy$)

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in the Kähler algebra in the real plane and, more specifically, on the strict harmonic ones. In this paper, we shall be oblivious to whether our integrands admit power expansions. That is a late development in the present context. It is left for interested parties to develop as in the standard complex variable calculus but with our simpler concepts.

The role of complex variable will again be played by $z \equiv x + ydxdy$ and by the relations

$$d\phi = \frac{1}{z}dy, \qquad d\rho = \frac{\rho}{z}dx. \tag{1}$$

An even smaller amount of Kähler algebra is now needed, but some minor concept in Kähler calculus is required [1].

# 2 Cauchy-Riemann and Cauchy-Goursat

## 2.1 Cauchy-Riemann equations

Clifford product will be indicated by juxtaposition. To avoid equivocity, we use the symbol $\partial$ for the operator

$$\partial \equiv dx\frac{\partial}{\partial x} + dy\frac{\partial}{\partial y}, \tag{2}$$

and define

$$d\alpha \equiv \partial \wedge \alpha, \quad \delta\alpha \equiv \partial \cdot \alpha. \tag{3}$$

We thus have

$$\partial\alpha = d\alpha + \delta\alpha. \tag{4}$$

$d\alpha$ is the standard exterior derivative. When $\alpha$ is a 1-form, $\delta\alpha$ is the scalar divergence.

If $u$ and $v$ are differentiable, $\partial(u + vdxdy)$ exists. A differential form that satisfies $\partial\alpha = 0$ is called strict harmonic. Denote Strict Harmonic edifs as $shedifs$. These admit multiplicative inverses and satisfy the Cauchy-Riemann relations:

$$\partial w = 0 \Longleftrightarrow \quad du = -\delta(vdxdy) \quad \Longleftrightarrow \quad u_{,x} = v_{,y}\,;\; u_{,y} = -v_{,x}\,. \tag{5}$$

Functions of edifs, and of shedifs in particular, can be defined as is usual in Clifford algebra, and thus as in the calculus of complex variable. Polynomial, rational, exponential, trigonometric, hyperbolic and logarithmic functions of shedifs are themselves shedifs.

## 2.2 Valuations and Cauchy-Goursat theorem

We replace integration $\int_c f(z)dz$ on a curve of the complex plane with valuation of $f(x+ydxdy)$ on a curve $c$ of the real plane. The valuation $\langle w\rangle_c$ of an edif $w$ on a curve $c$ is defined as the edif

$$\langle w\rangle_c \equiv \left[\int_c wdx\right] + dxdy\left[\int_c wdy\right], \tag{6}$$

with momentary use of square brackets for emphasis. Easy calculations yield

$$\langle w\rangle_c = \int_c (udx - vdy) \; + \; dxdy\int_c (udy + vdx). \tag{7}$$

The integrability conditions for these integrals to not depend on $c$ but only on the end points of the curve are the Cauchy-Riemann relations. Thus potentials

$$U = \int udx - vdy, \qquad\qquad V = \int (udy + vdx) \tag{8}$$

exist, which imply the existence of "valuation potentials" of shedifs,

$$\langle w\rangle = U + Vdxdy. \tag{9}$$

The valuation potential is determined only up to an additive constant shedif. It follows that the valuation of a shedif on a closed curve on those domains is zero. This is the Cauchy-Goursat theorem for shedifs.

$W$ is a shedif since

$$dU = udx - vdy, \quad dV = udy + vdx, \tag{10}$$

which implies

$$U_{,x} = u = V_{,y} \qquad\qquad U_{,y} = -v = V_{,x}\,. \tag{11}$$

The valuation plays the role played by integration in the calculus of complex variable.

In domains that are not simply connected, we surround the poles enclosed by closed curves $C$ with equally oriented circles $c_i$, all of them with the same orientation as $C$ and containing one and only one pole each. We then have

$$\langle w\rangle_C = \sum_i \langle w\rangle_{c_i}\,. \tag{12}$$

## 2.3 Rationale for the introduction of the concept of valuation

Consider the integral with integer $n$

$$\oint \frac{dz}{(z-z_0)^{n+1}} \tag{13}$$

in the standard calculus of complex variable. It is $2\pi i$ for $n = 0$, and 0 otherwise. Let us rewrite it as

$$\oint \frac{\frac{1}{(z-z_0)^n}}{z-z_0} dz \tag{14}$$

for potential integral as in [2] of integrals of the form

$$\oint \frac{f(z)}{z-z_0} dx \tag{15}$$

(Recall $dz = d(x + ydxdy) = dx$). Here also, the integral (15) has meaning. With $f(z)$ equal to $(z-z_0)^{-n}$, its value is zero, also for $n = 0$. It is not equivalent to the integral (13) in the standard calculus of complex variable. As for the restricted Cauchy's integral formula, it is not applicable in this case because $(z-z_0)^{-n}$ is not a differential $2-$form, not even for $n = 0$.

The role of (13) is now played by

$$\left\langle \frac{1}{(z-z_0)^{n+1}} \right\rangle_C . \tag{16}$$

As per definition (6), we have

$$\langle w \rangle_c \equiv \int_c \frac{1}{(z-z_0)^{n+1}} dx + dxdy \int_c \frac{1}{(z-z_0)^{n+1}} dy. \tag{17}$$

The first integral here is zero. and so is the second one except for $n = 0$, in which case (17), and thus (16), is $2\pi dxdy$. This conclusion about the value of (17) can be obtained as a direct consequence of the theorem of residues [2]. However, if we want to develop this approach independently of the previous one, we look at (17) from the perspective of the multivariable calculus. For integration on circles centered at $(x_0, y_0)$, we take into account that

$$\frac{1}{(z-z_0)^n} = \rho^{-n}(\cos n\phi - dxdy \sin n\phi), \tag{18}$$

and the stated result $2\pi dxdy$ again follows.

# 3 Cauchy's formulas

## 3.1 Cauchy's integral formula

We no longer face the restriction of the previous paper for this formula. But, unlike what was the case in the previous paper, we cannot approach the theorem in the very expeditiously way available through the theorem of residues, not yet derived here.

Let $f(z)$ be a shedif on a simply connected region of the real plane. The limit at $z = z_0$ of $f(z) - f(z_0)$ then is zero since the scalar and 2-form parts, $u$ and $v$, of $f(z)$ have derivatives and are, therefore, continuous. The edif

$$\frac{f(z) - f(z_0)}{z - z_0} \tag{19}$$

has a first order pole at $z_0$. By virtue of that continuity, we have

$$\left\langle \frac{f(z)}{z - z_0} \right\rangle_C = \left\langle \frac{f(z_0)}{z - z_0} \right\rangle_C, \tag{20}$$

as follows from (6). The right hand side is

$$\oint \frac{f(z_0)}{z - z_0} dx \ + \ dxdy \oint \frac{f(z_0)}{z - z_0} dy. \tag{21}$$

It can be computed explicitly on circles centered at $z_0$. Since $f(z_0)dx$ equals $u_0 dx - v_0 dy$, and $f(z_0)dy$ equals $u_0 dy + v_0 dx$, we get

$$\left\langle \frac{f(z_0)}{z - z_0} \right\rangle_C = \oint \frac{-v_0}{z - z_0} dy \ + \ dxdy \oint \frac{u_0}{z - z_0} dy, \tag{22}$$

where we have used that the integrations over $dx$ vanish because they are integrations over $d\rho$ by virtue of (1). Hence, finally,

$$\left\langle \frac{f(z)}{z - z_0} \right\rangle_C = 2\pi dxdy(u_0 + v_0 dxdy) = 2\pi dxdy f(z_0) \tag{23}$$

This equation is also given the alternative form

$$f(z_0) = \frac{1}{2\pi dxdy} \left\langle \frac{f(z)}{z - z_0} \right\rangle_C. \tag{24}$$

As an example, let $C$ denote the circle of radius 1 centered at the pole $z=0$. Since $z=\frac{\pi}{2}$ lies outside that circle, we have

$$\left\langle \frac{1}{z(z-\frac{\pi}{2})} \right\rangle_C = \left\langle \frac{\frac{1}{z-\frac{\pi}{2}}}{z} \right\rangle_C = 2\pi dxdy \frac{1}{-\frac{\pi}{2}} = -4\pi dxdy, \tag{25}$$

which is an integral for which there was no correspondence in the Weierstrass approach.

## 3.2 Co-valuations

The co-valuation of a shedif, $w = u + vdxdy$ ($\partial w = 0$) is defined as $\partial w / \partial x$,

$$\frac{\partial w}{\partial x} = u,_x + v,_x dxdy = v,_y dydy + v,_x dxdy = dvdy, \tag{26}$$

and also

$$\frac{\partial w}{\partial x} \equiv u,_x dxdx + u,_y dydx = dudx. \tag{27}$$

Clearly $\partial z / \partial x = 1$. From (7), eliminating the subscript because the result is curve independent, we get

$$\frac{\partial}{\partial x} \langle w \rangle = \int_c (u,_x dx - v,_x dy) \ + \ dxdy \int_c (u,_x dy + v,_x dx). \tag{28}$$

Using the Cauchy-Riemann conditions, we further obtain

$$\frac{\partial}{\partial x} \langle w \rangle = w. \tag{29}$$

Up to an arbitrary constant differential form, the equalities

$$\left\langle \frac{\partial w}{\partial x} \right\rangle = w = \frac{\partial}{\partial x} \langle w \rangle , \tag{30}$$

are further completed.

## 3.3 Cauchy's integral formula for derivatives

Rewrite (24) as

$$f(z) = \frac{1}{2\pi dxdy} \left\langle \frac{f(\zeta)}{\zeta - z} \right\rangle_C . \tag{31}$$

Clearly

$$\frac{\partial f(z)}{\partial x} = \frac{1}{2\pi dxdy} \left\langle \frac{\partial}{\partial x} \frac{f(\zeta)}{\zeta - z} \right\rangle_C = \frac{1}{2\pi dxdy} \left\langle \frac{f(\zeta)}{(\zeta - z)^2} \right\rangle_C . \tag{32}$$

Successive application yields

$$\frac{\partial^n f(z)}{\partial x^n} = \frac{n!}{2\pi dxdy} \left\langle \frac{f(\zeta)}{(\zeta - z)^{n+1}} \right\rangle . \tag{33}$$

This formula is equivalent to formula (42) of the previous paper, but without the restriction that $f(z)$ be a differential $2-$form.

As an example, we have

$$\left\langle \frac{1}{(z^2+1)^2} \right\rangle_C = \left\langle \frac{\frac{1}{(z+dxdy)^2}}{(dxdy - z)^2} \right\rangle_C . \tag{34}$$

for valuation around the pole $(0,1)$, i.e. $z_0 = dxdy$. We identify the values $n = 1$ and $f(z) = (z + dxdy)^{-2}$ for application of (33)

$$\left\langle \frac{1}{(z^2+1)^2} \right\rangle_C = 2\pi dxdy \left[ \frac{\partial}{\partial x} \frac{1}{(z + dxdy)^2} \right]_{z=dxdy} = \frac{\pi}{2}. \tag{35}$$

# 4 Concluding remarks

Once Cauchy's integral formulas for this formalism have been developed, the arguments leading to the Laurent series and the theorem of residues, as well as to the obtaining of residues for poles of higher order totally parallels the standard treatment of the same subjects in the standard calculus of complex variables. Being it trivial for the cognoscenti, we do not need to that here.

The Weierstrass and Cauchy's points of view have their respective merits. Once the difference in approach has been made clear, the following meshing of the two developments seems to us to be the most appropriate. Start with Weierstrass to obtain the theorem of residues. Then, instead of tackling Cauchy's integral formulas, go to the Laurent series as in that paper [2]. Only then address the restricted Cauchy's integral formulas, the restriction being of the essence of that approach, which is real in the most strict sense of the word. But the restriction also constitutes motivation for the present,

more general approach, with its new concepts of shedifs, valuation and co-valuation. At this points, researchers can decide whether it is in their best interest or their students to consider again the Laurent series, the theorem of residues and the obtaining of residues for poles of higher order

On the opposite, conservative end, there will be also those who think that it is not yet the time to teach their students this part of the real calculus. We dear say, however, that they would be making their students a big favor by teaching their students the theorem of residues as per the previous paper, even if that is all that they do before proceeding to teach the standard calculus of complex variable. It may spark their imagination and eventually become fervent seekers of the best way of do mathematics. The may find like this author did —alas too late— that the calculus of differential forms constitutes the language of choice for the mathematical needs of theoretical physicists.

# References


[1] Kähler, E.: Der innere Differentialkalkül. Rendiconti di Matematica, 21, 425-523 (1962).

[2] Vargas, J.G.: Real Version of Calculus of Complex Variable: Weierstrass Point of View.